A Critical Comparison of the 4-Intersection and 9-Intersection Models for Spatial Relations: Formal Analysis*

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Abstract

Two formalisms for binary topological spatial relations are compared for their expressive power. The *4-intersection* considers the two objects' interiors and boundaries and analyzes the intersections of these four object parts for their content (i.e., emptiness and non-emptiness). The *9-intersection* adds to the 4-intersection the intersections with the two objects' complements. The major results are (1) for objects with co-dimension 0, the 4-intersection and the 9-intersection with the content invariant provide the same results; and (2) for objects with co-dimension > 0, the 9-intersection with the content invariant provides more details than the 4-intersection. These additional details are crucial to determine when two objects are equal. It is also demonstrated that the additional details can provide crucial information when specifying the semantics of spatial relations in GIS query languages.

Introduction

During the last three years, the formal description of spatial relations has received unprecedented attention in the GIS arena. The focus of many investigations was on a particular formalism to represent *topological relations* (Egenhofer and Franzosa 1991; Herring 1991; Pigot 1991; Hadzilacos and Tryfona 1992; Hazelton *et al.* 1992; Clementini *et al.* 1993; Cui *et al.* 1993; Wazinski 1993). Complementary activities in the area of cardinal directions (Peuquet and Ci-Xiang 1987; Frank 1992; Freksa 1992; Papadias and

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Sellis 1992; Jungert 1992) exist, however, unlike the studies of topological relations, formalizations of cardinal directions are based on a diversity of models. This paper focuses on the two primary models used for binary topological relations, the 4-intersection and the 9-intersection, which is an extension of the 4-intersection.

The initial model for binary topological relations, developed for two 2-dimensional objects embedded in $\mathbb{R}^2$, compared the boundaries and interiors of the two objects and classified the relations by whether the intersections of these four parts were empty or not (Egenhofer 1989; Egenhofer and Herring 1990; Egenhofer and Franzosa 1991). This model is called the *4-intersection.* An extension of the 4-intersection includes also the intersections with the exteriors, and allows for the identification of more detailed relations, particularly if one or both objects are embedded in higher-dimensional spaces, such as the topological relation between a line and an areal object or the topological relation between two lines in $\mathbb{R}^2$ (Egenhofer and Herring 1991). This model is called the *9-intersection.*

The need for the more extensive 9-intersection has been questioned by several researchers who have tried to model line-region and line-line relations in $\mathbb{R}^2$ just with the 4-intersection (Svensson and Zhexue 1991; Hadzilacos and Tryfona 1992; Hazelton *et al.* 1992; Clementini *et al.* 1993). This paper demonstrates that the 4-intersection and 9-intersection reveal the same results only if both objects are n-dimensional and embedded in $\mathbb{R}^n$ such that different between the dimensions of the embedding space and the objects is 0. These objects are said to have co-dimension 0. For all other configurations with co-dimension > 0, such as the relations between a line and a region in $\mathbb{R}^2$ or the relations between two lines in $\mathbb{R}^2$, it is shown that the 9-intersection distinguishes among topological relations that would be considered the same using the 4-intersection model.

The remainder of this paper is structured as follows: The next section briefly reviews the 4-intersection and the 9-intersection models for topological relations. Then the consequences of using the 4-intersection or 9-intersection are elaborated for line-region and line-line relations in $\mathbb{R}^2$. A discussion of using alternatives to the boundary/interior 4-intersections completes the comparison of different models for binary topological relations. The conclusions provide a concise summary of the results and their importance.

Models for Topological Relations

4-Intersection

Binary topological relations between two objects, A and B, are defined in terms of the four intersections of A's boundary (∂A) and interior (A°) with the boundary (∂B) and interior (B°) of B (Egenhofer and Franzosa 1991). This model is concisely represented by a 2×2-matrix, called the *4-intersection.*

$$\Im_4(A,B) = \begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B \\ \partial A \cap B^\circ & \partial A \cap \partial B \end{pmatrix} \quad (1)$$

Topological invariants of these four intersections, i.e., properties that are preserved under topological transformations, are used to categorize topological relations. Examples of topological invariants, applicable to the 4-intersection, are the content (i.e., emptiness or non-emptiness) of a set, the dimension, and the number of separations (Franzosa and Egenhofer 1992). The content invariant is the most general criterion as other invariants can be considered refinements of non-empty intersections. By considering the values empty ($\varnothing$) and non-empty ($\neg\varnothing$) for the four intersections, one can distinguish $2^4 = 16$ binary topological relations. Eight of these sixteen relations can be realized for homogeneously

2-dimensional objects with connected boundaries, called *regions*, if the objects are embedded in $\mathbb{R}^2$ (Egenhofer and Herring 1990) (Figure 1).

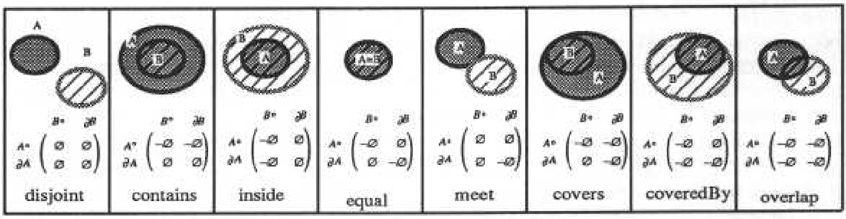


Figure 1: Examples of the eight topological relations between two regions in $\mathbb{R}^2$.

Also eight topological relations can be found between two *lines* in $\mathbb{R}^1$ (Pullar and Egenhofer 1988) (Figure 2).The latter set of relations corresponds to Allen's interval relations (Allen 1983) if the order of $\mathbb{R}^1$ is disregarded. With the exception of *overlap*, the two sets of 4-intersections for region-region relations in $\mathbb{R}^2$ and line-line relations in $\mathbb{R}^1$ are identical. The difference is due to the fact that regions have *connected* boundaries, while lines have *disconnected* boundaries; therefore, for a region whose boundary intersects with the other region's interior *and* exterior, its boundary must also intersect with the other region's boundary. This conclusion cannot be drawn for two lines because their boundaries are disconnected.

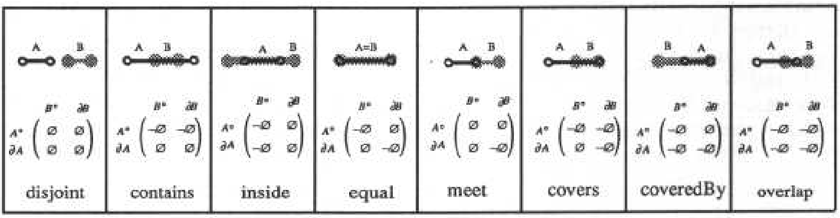


Figure 2: Examples of the eight topological relations between two lines in $\mathbb{R}^1$.

9-Intersection

The 4-intersection model is extended by considering the location of each interior and boundary with respect to the other object's exterior; therefore, the binary topological relation between two objects, A and B, in $\mathbb{R}^2$ is based upon the intersection of A's interior (A°), boundary (∂A), and exterior (A^-) with B's interior (B°), boundary (∂B), and exterior (B^-). The nine intersections between the six object parts describe a topological relation and can be concisely represented by a 3×3-matrix $\Im_9$, called the *9-intersection*.

$$\Im_9(A,B)=\begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^- \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap B^\circ & A^- \cap \partial B & A^- \cap B^- \end{pmatrix} \tag{2}$$

In analogy to the 4-intersection, each intersection will be characterized by a value empty ($\varnothing$) or non-empty ($\neg\varnothing$), which allows one to distinguish $2^9 = 512$ different configurations. Only a small subset of them can be realized between two object in $\mathbb{R}^2$.

The relations that can be realized depend on particular topological properties of the objects involved and their relationship to the embedding space. For example, the boundary of a spatial region in $\mathbb{R}^2$ is a Jordan Curve (separating the interior from the exterior). On the other hand, the boundary of a simple line consists of two nodes, and unlike a region's boundary in $\mathbb{R}^2$, a line's boundary in $\mathbb{R}^2$ does not separate the interior from the exterior. These topological properties of the objects have to be considered when investigating which empty/non-empty 9-intersection can be realized. For this goal, we formalized for each combination of regions and lines embedded in $\mathbb{R}^2$ a set of properties as conditions for binary topological relations, that must hold between the parts of the two objects (Egenhofer and Herring 1991). These properties can be expressed as consistency constraints in terms of the 9-intersection, such that by successively eliminating from the set of 512 relations those relations that violate a consistency constraint, one retains the candidates for those 9-intersections that can be realized for the particular spatial data model (Egenhofer and Sharma 1993). The existence of these relations is then proven by finding geometric interpretations for the corresponding 9-intersections.

Existing 9-Intersections Between Two Regions in $\mathbb{R}^2$: With the 9-intersection, the same set of region-region relations can be found as for the 4-intersection (Egenhofer and Franzosa 1991). No additional relations due to the consideration of exterior-intersections are possible.

Existing 9-Intersections Between Two Lines in $\mathbb{R}^2$: As expected, the 9-intersection reveals the same number of line-line relations in $\mathbb{R}^1$ as the 4-intersection; however, in $\mathbb{R}^2$, the 9-intersection identifies another 25 relations for relations between two simple lines (i.e., lines with exactly two end points) (Egenhofer 1993). Another 21 relations are found if the lines can be branched so that they have more than two end points (Egenhofer and Herring 1991).

Existing 9-Intersections Between A Line and a Region in $\mathbb{R}^2$: With the 9-intersection, 19 topological relations between a simple line and a region in $\mathbb{R}^2$ can be found (Mark and Egenhofer 1992), and a 20th configuration if the line is branched (Egenhofer and Herring 1991).

Need for 9-Intersection

Since the realization of existing topological relations in both models is based on particular topological properties of the objects and the relationship to the embedding space, the following generalizations can be made:

- If the two objects are simply connected, their boundaries form Jordan Curves (or the corresponding configurations in higher-dimensional spaces), and the objects have co-dimension 0, then the same eight topological relations can be realized as between two regions in $\mathbb{R}^2$. For example, the same relations as between two regions in $\mathbb{R}^2$ also exist between two volumes.
- If the two objects are simply connected, each boundary forms a separation, and the objects have co-dimension 0, then the same eight topological relations can be realized as between two lines in $\mathbb{R}^1$.

If the co-dimension constraint is relaxed such that one or both objects can be embedded in a higher-dimensional space, due to the greater degree of freedom, the objects may take additional configurations that are not represented by one of the relations between objects with co-dimension 0. For example, if two lines "cross," they have non-empty interior-interior intersections, while the other three intersections are empty. Such a 4-intersection cannot be realized for two lines in $\mathbb{R}^1$. On the other hand, some 4-intersections may have ambiguous geometric interpretations. From a practical point of view, there are certainly situations in which one would like to distinguish between them when querying a geographic database. Therefore, in general, a different model is necessary to account also for relations involving n-dimensional objects that are embedded in $\mathbb{R}^m$, $m > n$. Our focus is on $m = 2$ and its topological relations with objects of dimension $n = 1$—topological relations with points ($n = 0$) are trivial.

To analyze the differences between the two methods, the conceptual neighborhoods of each set of relations will be used. *Conceptual neighborhoods* organize a set of relations in a diagram such that similar relations are close to each other. The computational tool to identify conceptual neighborhoods is the topology distance (Egenhofer and Al-Taha 1992) which calculates the differences between two empty/non-empty 9-intersections. Pairs of relations with the least number of non-zero differences are considered to be conceptual neighbors. Conceptual neighborhoods are represented as a graph in which the relations are nodes and the conceptual neighbors are edges between the nodes.

Line-Region Relations

If one removes for each of these nineteen 9-intersections the entries of the exterior intersections, what remains is a 4-intersection based on boundary/interior intersections. A straightforward inspection reveals that only six of the 19 line-region relations are uniquely characterized by the 4-intersection. These relations are all located along the edge of the conceptual neighborhood diagram shown in Figure 3 (A1, B1, C1, C5, D5, and E5). The remaining thirteen cases can be grouped into five distinct groups, each having a characteristic 4-intersection: A2, A3, and A5, called the A-band; and the B-, C-, D-, and E-band with B2-B3, C2-C4, D2-D4, and E3-E4, respectively. The distinguishing factor among the configurations in each group is whether the interior or boundary of the line intersects the exterior of the region. Since the intersections with the exterior are not considered, the 4-intersection is the same for all configurations in any band.

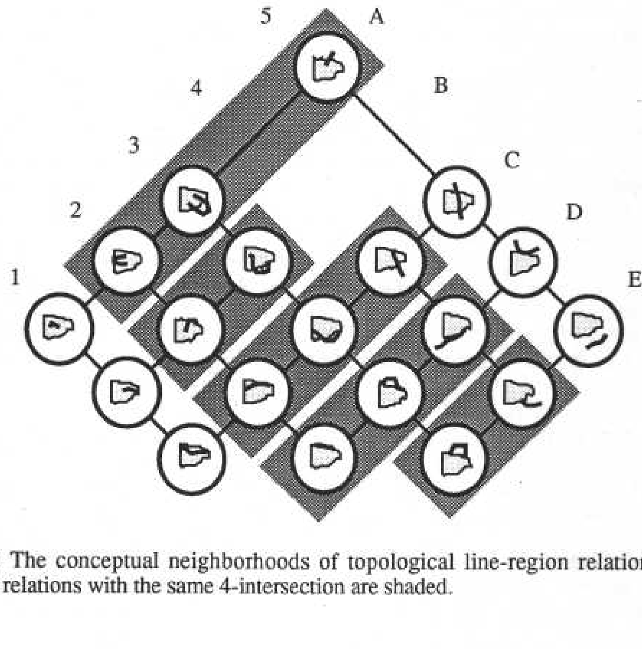
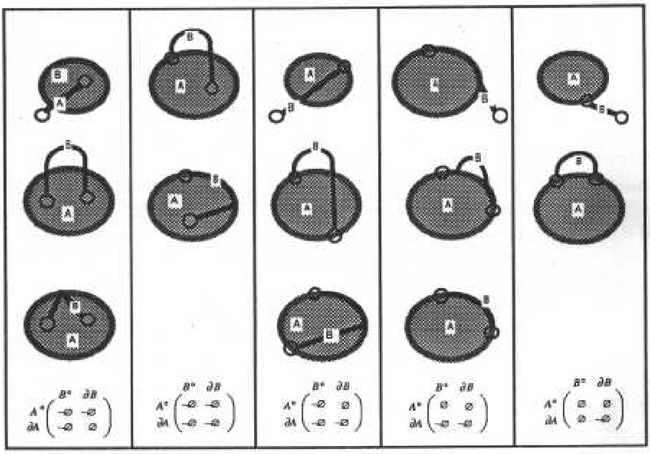


Figure 3: The conceptual neighborhoods of topological line-region relations in $\mathbb{R}^2$. Groups of relations with the same 4-intersection are shaded.

In the data model the interior, boundary, and exterior are distinct topological entities. Hence each configuration, in any group in Figure 4, is topologically distinct from the other configurations in the same group. That is, there is no topological transformation that converts one configuration to another. The 4-intersection, however, cannot distinguish between them and thus it is insufficient.



A B C D E

Figure 4: The five groups of line-region relations. Relations in each group have the same 4-intersection, but distinct 9-intersections.

The importance of the additional information available in the 9-intersection becomes more obvious when one investigates the meaning of certain spatial predicates. Spatial predicates are commonly used as selection criteria in GIS queries and in order to process such queries, the semantics of the terms have to be formalized. If one considers the line and the region to be a road and a park, respectively, then one may consider the meaning of the spatial predicate "enters" as "the line has to have parts inside and outside of the region." Based on the 9-intersection, the six configurations with non-empty interior-interior, interior-boundary, and interior-exterior intersections qualify for this constraint—A3, A5, B3, and C3, C4, C5. Therefore, such a definition of "enters" splits the shaded bands (A, B, and C), i.e., the configurations that cannot be distinguished by the 4-intersection. Based solely on the 4-intersection, this distinction would not have been possible, because the set of relations with non-empty interior-interior and interior-boundary intersections includes three configurations with non-empty interior-exterior intersections; therefore, using the 4-intersection or the 9-intersection as the underlying model to process such a query, one may get considerably different results, some of which would contradict the definition of the term "enters."

The question remains open whether the 4-intersection would be sufficient if one had particular knowledge about the objects' geometric properties such as whether the lines are straight or possibly curved, and whether the regions are convex or possibly concave. First, the knowledge of only one such geometric property does not influence the existence of topological relations. For example, if one fixes the shape of the line to be straight then the region can be deformed, where necessary, to a concave object so that all 19 relations can be realized. Likewise, if the region were fixed to be convex, one could bend the line so that all 19 relations can be realized. The case is, however, different if both objects are constrained.

Among the 19 line-region relations, only 11 can be found for a straight line and a convex region. The eight additional ones for curved lines or convex regions all fall in the range of relations that cannot be distinguished with the 4-intersection: they are A2 and A3, B2 and B3, C2 and C3, D3, and E3. With the exception of the relations in band D, there is a 1:1 mapping between the 4-intersection- and 9-intersection-relations for a straight line and a convex region (the relations in band B cannot be realized in either model). The two straight-line-convex-region-relations that cannot be distinguished are (1) a straight line is completely in the boundary of a convex region and (2) a straight line starts in the boundary following the boundary for a while, until it ends in the convex region's exterior.

Line-Line Relations

The 9-intersection characterization results in 33 distinct line-line relations in $\mathbb{R}^2$ (Egenhofer and Herring 1991). Since the 4-intersection can only characterize $2^4 = 16$ distinct relations, it is obvious that the 9-intersection provides a much finer resolution. Figure 5 shows a subset of the conceptual neighborhood of topological line-line relations in $\mathbb{R}^2$ and highlights the groups of those relations that would not be distinguished by the 4-intersection. There are 10 such groups containing between two and four relations. Six relations from the 4-intersection have exactly one corresponding 9-intersection relation.

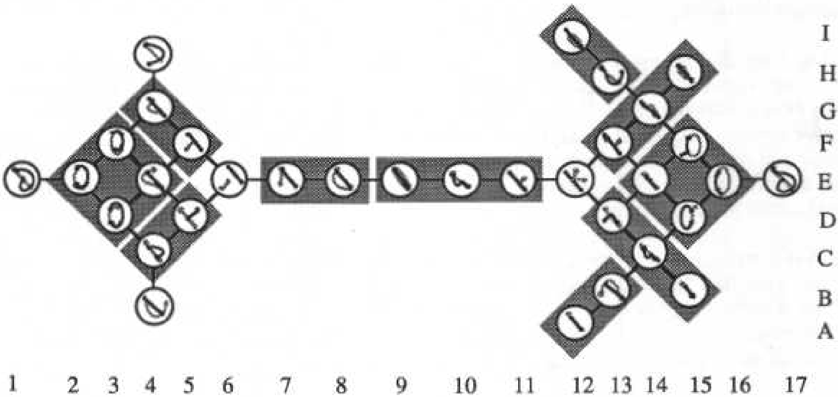


Figure 5: A subset of the conceptual neighborhood of topological line-line relations in $\mathbb{R}^2$ (not all links of topology distance 1 are depicted). Groups of relations with the same 4-intersection are shaded.

As a strongly motivating example for the need of the finer granularity of the 9-intersection, consider the topological relation *equal* (E9). Equal is part of a group with another two relations, all of which have the same 4-intersection (Figure 6), and not a singleton as one would expect. Using the 9-intersection, only the configuration in Figure 6a would be classified as an example of *equal*.

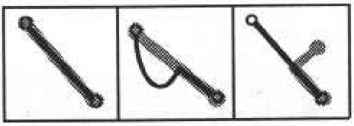


Figure 6: Examples of topological relations between two lines in $\mathbb{R}^2$ that have the same 4-intersection ($A^\circ \cap B^\circ = \neg\varnothing$; $A^\circ \cap \partial B = \varnothing$; $\partial A \cap B^\circ = \varnothing$; and $\partial A \cap \partial B = \neg\varnothing$), but different 9-intersections as their boundary-exterior and interior-exterior intersections differ.

Alternative 4-Intersections

The data model used here is based on concepts from point-set topology. A spatial region is a simply connected area whose boundary is a Jordan curve; therefore, it has three topologically distinct parts: the interior, boundary, and exterior. Since a region is a 2-dimensional object in a 2-dimensional space, specifying any one part completely determines the region and its other parts.

Based on this observation it appears reasonable to assume that topological relations between regions can be characterized by considering the intersections of any pair of parts, for example, boundary/exterior or interior/exterior, rather than only the boundary/interior intersections. To assess such alternatives, one has to determine whether the 4-intersection based on the boundary/interior intersections is equivalent to one based on boundary/exterior or interior/exterior intersections. If so, the characterization of topological relations would have to be the same in each case.

- A 4-intersection based on boundary/exterior intersections cannot express the distinction between the relations *meet* and *overlap*. The reason is that the only difference between meet and overlap is whether the interiors do not or do intersect, respectively. Since the intersections of interiors is not considered, the 4-intersections, for the configurations called meet and overlap in Figure 1, are exactly the same.
- Similarly, a 4-intersection based on interior/exterior intersections cannot express the distinction between the pairs of relations: *meet* and *disjoint*, *contains* and *covers*, *inside* and *coveredBy*, because the only difference in each case is whether the boundaries intersect or not. Since the intersection of boundaries is not considered, the 4-intersections are exactly the same.
- Finally, the alternatives of using a 4-intersection based on the closure—the union of the interior and boundary—in combination with the interior, boundary, or exterior reveal the same deficiencies as they cannot distinguish between overlap and covers/coveredBy, or overlap and meet.

The conclusion is therefore that only boundary and interior should be used for the 4-intersection in characterizing topological relationships between regions.

Conclusions

The two primary models of topological relations, the 4-intersection and the 9-intersection, were compared for their expressive powers. Table 1 summarizes the results of the numbers of relations that can be realized in each model for co-dimension 0. It was shown in this paper that for co-dimension 0 exactly the same relations can be realized with the 4- and 9-intersection.

co-dimension 0	**region**	**line**
region	$\Im_4$: 8 relations $\Im_9$: 8 relations	N/A
line	N/A	$\Im_4$: 8 relations $\Im_9$: 8 relations

Table 1: Number of binary topological relations that can be realized for regions and lines in co-dimension 0 with the 4-intersection and the 9-intersection.

The situation is quite different if the two objects are embedded in a higher-dimensional space (Table 2). The 9-intersection has a finer granularity to distinguish relations between a line and a region, and between two lines embedded in $\mathbb{R}^2$. The most crucial difference was found for line-line relations, where the 4-intersection applied to $\mathbb{R}^2$ does not provide a useful definition of an "equal" relation. On the other hand, the 9-intersection compensates this shortcoming.

co-dimension 1	**line**	*straight line*
region	$\Im_4$: 11 relations $\Im_9$: 19 relations	
convex region		$\Im_4$: *10 relations* $\Im_9$: *11 relations*
line	$\Im_4$: 16 relations $\Im_9$: 33 relations	
straight line		$\Im_4$: *11 relations* $\Im_9$: *11 relations*

Table 2: Number of binary topological relations that can be realized for regions and lines in co-dimension 1 with the 4-intersection and the 9-intersection.

The results have an impact on the implementation of spatial relations in a GIS. Although the 9-intersection is necessary to distinguish such details, not all nine intersections have to be calculated at all times when processing a query with such a topological relation. Most obvious is this in the case of the line-region relations, where all intersections between the line's exterior and the three parts of the region are non-empty, independent of the relation between the two objects; therefore, calculating these three intersections would not provide any information about the particular configuration.

The results of this paper must be considered in combination with results obtained from human-subject testing of topological relations (Mark and Egenhofer 1992). Initial studies of line-region configurations showed there that the differences in the distinctions made by the 9-intersection are sometimes crucial when humans select natural-language terminology to describe some spatial situations. Only if the present analysis is considered in the entirety of the *interplay* between formal mathematics and human-subjects testing, its significance will become obvious.

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References

J. F. Allen (1983) Maintaining Knowledge about Temporal Intervals. *Communications of the ACM* 26(11): 832-843.

E. Clementini, P. Di Felice, and P. van Oosterom (1993) A Small Set of Formal Topological Relationships Suitable for End-User Interaction. in: D. Abel and B. C. Ooi (Eds.), *Third International Symposium on Large Spatial Databases, SSD '93. Lecture Notes in Computer Science* 692, pp. 277-295, Springer-Verlag, New York, NY.

Z. Cui, A. Cohn, and D. Randell (1993) Qualitative and Topological Relationships in Spatial Databases. in: D. Abel and B. Ooi (Eds.), *Third International Symposium on Large Spatial Databases. Lecture Notes in Computer Science* 692, pp. 296-315, Springer-Verlag, New York, NY.

M. Egenhofer (1989) A Formal Definition of Binary Topological Relationships. in: W. Litwin and H.-J. Schek (Ed.), *Third International Conference on Foundations of Data Organization and Algorithms (FODO). Lecture Notes in Computer Science* 367, pp. 457-472, Springer-Verlag, New York, NY.

M. Egenhofer (1993) Definitions of Line-Line Relations for Geographic Databases, *IEEE Data Engineering* 16 (in press).

M. Egenhofer and K. Al-Taha (1992) Reasoning About Gradual Changes of Topological Relationships. in: A. Frank, I. Campari, and U. Formentini (Eds.), *Theories and Models of Spatio-Temporal Reasoning in Geographic Space. Lecture Notes in Computer Science* 639, pp. 196-219, Springer-Verlag, New York, NY.

M. Egenhofer and R. Franzosa (1991) Point-Set Topological Spatial Relations. *International Journal of Geographical Information Systems* 5(2): 161-174.

M. Egenhofer and J. Herring (1990) A Mathematical Framework for the Definition of Topological Relationships. *Fourth International Symposium on Spatial Data Handling*, Zurich, Switzerland, pp. 803-813.

M. Egenhofer and J. Herring (1991) *Categorizing Binary Topological Relationships Between Regions, Lines, and Points in Geographic Databases*. Technical Report, Department of Surveying Engineering, University of Maine, Orono, ME.

M. Egenhofer and J. Sharma (1993) Topological Relations between Regions in R^2 and Z^2. in: D. Abel and B. Ooi (Eds.), *Third International Symposium on Large Spatial Databases. Lecture Notes in Computer Science* 692, pp. 316-336, Springer-Verlag, New York, NY.

A. Frank (1992) Qualitative Spatial Reasoning about Distances and Directions in Geographic Space. *Journal of Visual Languages and Computing* 3(4): 343-371.

R. Franzosa and M. Egenhofer (1992) Topological Spatial Relations Based on Components and Dimensions of Set Intersections. *SPIE's OE/Technology '92—Vision Geometry*, Boston, MA, pp. 236-246.

C. Freksa (1992) Using Orientation Information for Qualitative Spatial Reasoning. in: A. Frank, I. Campari, and U. Formentini (Eds.), *Theories and Models of Spatio-Temporal*

Reasoning in Geographic Space. Lecture Notes in Computer Science 639, pp. 162-178, Springer-Verlag, New York, NY.

T. Hadzilacos and N. Tryfona (1992) A Model for Expressing Topological Integrity Constraints in Geographic Databases. in: A. Frank, I. Campari, and U. Formentini (Eds.), *Theories and Models of Spatio-Temporal Reasoning in Geographic Space. Lecture Notes in Computer Science* 639, pp. 252-268, Springer-Verlag, Pisa.

N. W. Hazelton, L. Bennett, and J. Masel (1992) Topological Structures for 4-Dimensional Geographic Information Systems. *Computers, Environment, and Urban Systems* 16(3): 227-237.

J. Herring (1991) The Mathematical Modeling of Spatial and Non-Spatial Information in Geographic Information Systems. in: D. Mark and A. Frank (Ed.), *Cognitive and Linguistic Aspects of Geographic Space*. pp. 313-350, Kluwer Academic Publishers, Dordrecht.

E. Jungert (1992) The Observer's Point of View: An Extension of Symbolic Projections. in: A. Frank, I. Campari, and U. Formentini (Eds.), *Theories and Models of Spatio-Temporal Reasoning in Geographic Space. Lecture Notes in Computer Science* 639, pp. 179-195, Springer-Verlag, New York, NY.

D. Mark and M. Egenhofer (1992) An Evaluation of the 9-Intersection for Region-Line Relations. *GIS/LIS '92*, San Jose, CA, pp. 513-521.

D. Papadias and T. Sellis (1992) Spatial Reasoning Using Symbolic Arrays. in: A. Frank, I. Campari, and U. Formentini (Eds.), *Theories and Models of Spatio-Temporal Reasoning in Geographic Space. Lecture Notes in Computer Science* 639, pp. 153-161, Springer-Verlag, New York, NY.

D. J. Peuquet and Z. Ci-Xiang (1987) An Algorithm to Determine the Directional Relationship Between Arbitrarily-Shaped Polygons in the Plane. *Pattern Recognition* 20(1): 65-74.

S. Pigot (1991) Topological Models for 3D Spatial Information Systems. *Autocarto 10*, Baltimore, MD, pp. 368-392.

D. Pullar and M. Egenhofer (1988) Towards Formal Definitions of Topological Relations Among Spatial Objects. in: D. Marble (Ed.), *Third International Symposium on Spatial Data Handling*, Sydney, Australia, pp. 225-242.

P. Svensson and H. Zhexue (1991) Geo-SAL: A Query Language for Spatial Data Analysis. in: O. Günther and H.-J. Schek (Eds.), *Advances in Spatial Databases—Second Symposium, SSD '91. Lecture Notes in Computer Science* 525, pp. 119-140, Springer-Verlag, New York, NY.

P. Wazinski (1993) Graduated Topological Relations. Technical Report 54, University of the Saarland, Saarbrücken, Germany.